

POLYNOMIALS REPRESENTING EYNARD-ORANTIN INVARIANTS

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ABSTRACT. The Eynard-Orantin invariants of a plane curve are multilinear differentials on the curve. For a particular class of genus zero plane curves these invariants can be equivalently expressed in terms of simpler expressions given by polynomials obtained from an expansion of the Eynard-Orantin invariants around a point on the curve. This class of curves contains many interesting examples.

1. INTRODUCTION

As a tool for studying enumerative problems in geometry Eynard and Orantin [3] associate multilinear differentials to any compact Riemann surface C equipped with two meromorphic functions x and y with the property that the branch points of x are simple and the map

$$\begin{aligned} C &\rightarrow \mathbb{C}^2 \\ p &\mapsto (x(p), y(p)) \end{aligned}$$

is an immersion. For every $(g, n) \in \mathbb{Z}^2$ with $g \geq 0$ and $n > 0$ they define a multilinear differential, i.e. a tensor product of meromorphic 1-forms on the product C^n , notated by $\omega_n^g(p_1, \dots, p_n)$ for $p_i \in C$. When $2g - 2 + n > 0$, $\omega_n^g(p_1, \dots, p_n)$ is defined recursively in terms of local information around branch points of x of $\omega_n^{g'}(p_1, \dots, p_n)$ for $2g' - 2 + n' < 2g - 2 + n$. A dilaton relation between ω_{n+1}^g and ω_n^g can be applied in the $n = 0$ case to define $F^g (= \omega_0^g)$ for $g \geq 2$, known as the symplectic invariants of the curve. The invariant F^g recursively uses all $\omega_n^{g'}$ with $g' + n \leq g + 1$.

In principle F^g and ω_n^g can be calculated explicitly from the recursion relations defining them, and implemented on a computer. In practice, the expressions obtained this way are unwieldy and computable only for small g and n . The main aim of this paper is to express ω_n^g in a simpler form—essentially its inverse discrete Laplace transform—and to develop methods to calculate general formulae for F^g in some examples.

We consider the Eynard-Orantin invariants of the class of genus zero curves for which x is two-to-one and has two branch points. The case of one branch point $x = z^2$ is dealt with in [3]. One can parametrise the domain so that the curve can be written:

$$(1) \quad C = \begin{cases} x = a + b(z + 1/z) \\ y = y(z) \end{cases}$$

for constants a, b and any rational function $y(z)$ with $y'(\pm 1) \neq 0$. The definition of ω_n^g and its properties requires the Riemann surface to be compact. Nevertheless, by taking sequences of compact Riemann surfaces one can extend the definition to allow $y(z)$ to be any analytic function defined on a domain in \mathbb{C} containing $z = \pm 1$, e.g. $y(z) = \ln(z)$ defined on the complement of the negative imaginary axis.

Theorem 1. *For the plane curve C defined in (1) and $2g - 2 + n > 0$, $\omega_n^g(z_1, \dots, z_n)$ has an expansion around $\{z_i = 0\}$ given by*

$$(2) \quad \omega_n^g(z_1, \dots, z_n) = \frac{d}{dz_1} \dots \frac{d}{dz_n} \sum_{b_i > 0} N_n^g(b_1, \dots, b_n) z_1^{b_1} \dots z_n^{b_n} dz_1 \dots dz_n$$

where N_n^g is a quasi-polynomial in the b_i^2 of homogeneous degree $3g - 3 + n$, dependent on the parity of the b_i and is symmetric in all variables of the same parity.

The parity dependence means that there exists polynomials $N_{n,k}^g(b_1, \dots, b_n)$ for $k = 1, \dots, n$ such that $N_n^g(b_1, \dots, b_n)$ decomposes

$$N_n^g(b_1, \dots, b_n) = N_{n,k}^g(b_1, \dots, b_n), \quad k = \text{number of odd } b_i.$$

The polynomials representing N_n^g are simpler expressions than $\omega_n^g(z_1, \dots, z_n)$ and can have meaning themselves. When $x = z + 1/z$ and $y = z$, N_n^g arises as a solution of a Hurwitz problem [7, 8], with typical expression $N_{4,0}^0 = \frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2) - 1$ (while the much larger expression ω_4^0 can be expressed as the sum of 32 rational functions.) The case $x = z + 1/z$ and $y = \ln z$ arises when studying partitions and the Plancherel measure [2], with $N_{4,0}^0 = \frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2)$.

Theorem 2. *The coefficients of the top homogeneous degree terms in the polynomial $N_{n,k}^g(b_1, \dots, b_n)$, defined above, can be expressed in terms of intersection numbers of tautological line bundles over the moduli space $L_i \rightarrow \mathcal{M}_{g,n}$. For $\sum_i \beta_i = 3g - 3 + n$, the coefficient v_β of $\prod b_i^{2\beta_i}$ is*

$$v_\beta = \frac{y'(1)^{2-2g-n} + (-1)^k y'(-1)^{2-2g-n}}{x''(1)^{2g-2+n} 2^{3g-3+n} \beta!} \int_{\mathcal{M}_{g,n}} c_1(L_1)^{\beta_1} \dots c_1(L_n)^{\beta_n}$$

The invariants ω_n^g satisfy string and dilaton equations—see Section 4—which enable one to express N_{n+1}^g recursively in terms of N_n^g . They are most easily expressed when $y(z)$ is a monomial. The following two theorems apply to the polynomials $N_{n,k}^g$ although we abuse notation and simply write N_n^g .

Theorem 3. *The polynomials associated to the plane curves*

$$x = z + 1/z, \quad y = z^m/m, \quad m = 1, 2, \dots$$

satisfy the following recursion relations.

$$(3) \quad N_{n+1}^g(m, b_1, \dots, b_n) = \sum_{j=1}^n \sum_{k=1}^{b_j} k N_n^g(b_1, \dots, b_n) |_{b_j=k}$$

$$(4) \quad (m+1)N_{n+1}^g(m+1, b_1, \dots, b_n) + (m-1)N_{n+1}^g(m-1, b_1, \dots, b_n) \\ = 2m \sum_{j=1}^n \sum_{k=1}^{b_j} k N_n^g(b_1, \dots, b_n) |_{b_j=k} - m \sum_{j=1}^n b_j N_n^g(b_1, \dots, b_n)$$

$$(5) \quad (m-2)N_{n+1}^g(m-2, b_1, \dots, b_n) - 2mN_{n+1}^g(m, b_1, \dots, b_n) \\ + (m+2)N_{n+1}^g(m+2, b_1, \dots, b_n) = m \sum_{j=1}^n \sum_{k=1 \pm b_j} k N_n^g(b_1, \dots, b_n)|_{b_j=k}$$

$$(6) \quad N_{n+1}^g(m+1, b_1, \dots, b_n) - N_{n+1}^g(m-1, b_1, \dots, b_n) = m(2g-2+n)N_n^g(b_1, \dots, b_n)$$

Corollary 1. *The symplectic invariants of the curve $x = z+1/z$, $y = z^m/m$ satisfy*

$$F^g = \frac{1}{m(2g-2)} (N_1^g(m+1) - N_1^g(m-1))$$

for $g \geq 2$.

The polynomials corresponding to $y = \ln z$ satisfy recursions obtained by setting $m = 0$ into (3) and $((4) - (6))/m$.

Theorem 3 is a special case of the following more general theorem that applies to any analytic function $y(z)$ defined on a domain in \mathbb{C} containing $z = \pm 1$ expanded as $y(z) \sim \sum (a_k + zb_k)(1-z^2)^k$. For example $y(z) = \ln z \sim \sum \frac{(1-z^2)^k}{-2k}$.

First we need the following notation. Define the operator \mathcal{D} on functions by

$$\mathcal{D}f(n) = f(n+1).$$

Further, define $\mathcal{D}\{f(n)\}_{n=a} = f(a+1)$. As usual, for a polynomial $p(z) = \sum p_i z^i$, define $p(\mathcal{D})\{f(n)\}_{n=a} = \sum p_i f(a+i)$. Note that $\mathcal{D} - I$ is a discrete derivative and $y(\mathcal{D}) \sim \sum (a_k + zb_k)(1-\mathcal{D}^2)^k$ is a sum over powers of the discrete derivative $\mathcal{D}^2 - I$. Put $b_S = (b_1, \dots, b_n)$ and $|b_S| = b_1 + \dots + b_n$.

Theorem 4. *For the plane curve $x = z + 1/z$, $y = y(z)$*

$$(7) \quad \mathcal{D}y(\mathcal{D}) \{mN_{n+1}^g(m, b_S)\}_{m=-1} = \sum_{j=1}^n \sum_{\substack{k=1 \\ k \neq b_j(2)}}^{b_j} k N_n^g(b_S)|_{b_j=k}$$

$$(8) \quad (1 + \mathcal{D}^2)y(\mathcal{D}) \{mN_{n+1}^g(m, b_S)\}_{m=-1} = 2 \sum_{j=1}^n \sum_{\substack{k=1 \\ k \equiv b_j(2)}}^{b_j} k N_n^g(b_S)|_{b_j=k} - |b_S| N_n^g(b_S)$$

$$(9) \quad (1 - \mathcal{D}^2)^2 y(\mathcal{D}) \{mN_{n+1}^g(m, b_S)\}_{m=-2} = \sum_{j=1}^n \sum_{k=1 \pm b_j} k N_n^g(b_S)|_{b_j=k}$$

$$(10) \quad (1 - \mathcal{D}^2)y(\mathcal{D}) \{N_{n+1}^g(m, b_S)\}_{m=-1} = (2 - 2g - n)N_n^g(b_S)$$

Although y may not be a polynomial the left hand sides of (7) - (10) are finite sums, since large enough powers of a discrete derivative vanish on the quasi-polynomial N_{n+1}^g . See Section 2.

Corollary 2. *The symplectic invariants of the curve $x = z + 1/z$, $y = y(z)$ satisfy*

$$F^g = \frac{1}{2-2g} (1 - \mathcal{D}^2)y(\mathcal{D}) \{N_1^g(m)\}_{m=-1}$$

for $g \geq 2$.

The definition of the Eynard-Orantin invariants is given in Section 2. The proofs of Theorems 1 and 2 are in Section 3 and the proof of Theorems 3 and 4 is in Section 3. Section 5 contains examples.

2. EYNARD-ORANTIN INVARIANTS.

For every $(g, n) \in \mathbb{Z}^2$ with $g \geq 0$ and $n > 0$ the Eynard-Orantin invariant of a plane curve C is a multilinear differential $\omega_n^g(p_1, \dots, p_n)$, i.e. a tensor product of meromorphic 1-forms on the product C^n , where $p_i \in C$. When $2g - 2 + n > 0$, $\omega_n^g(p_1, \dots, p_n)$ is defined recursively in terms of local information around the poles of $\omega_{n'}^{g'}(p_1, \dots, p_n)$ for $2g' + 2 - n' < 2g - 2 + n$. Equivalently, the $\omega_{n'}^{g'}(p_1, \dots, p_n)$ are used as kernels on the Riemann surface to integrate against. This is a familiar idea, the main example being the Cauchy kernel which gives the derivative of a function in terms of the bilinear differential $dwdz/(w - z)^2$ as follows

$$f'(z)dz = \operatorname{Res}_{w=z} \frac{f(w)dwdz}{(w - z)^2} = - \sum_{\alpha} \operatorname{Res}_{w=\alpha} \frac{f(w)dwdz}{(w - z)^2}$$

where the sum is over all poles α of $f(w)$.

The Cauchy kernel generalises to a bilinear differential $B(w, z)$ on any Riemann surface C given by the meromorphic differential $\eta_w(z)dz$ unique up to scale which has a double pole at $w \in C$ and all A -periods vanishing. The scale factor can be chosen so that $\eta_w(z)dz$ varies holomorphically in w , and transforms as a 1-form in w and hence it is naturally expressed as the unique bilinear differential on C

$$B(w, z) = \eta_w(z)dwdz, \quad \oint_{A_i} B = 0, \quad B(w, z) \sim \frac{dwdz}{(w - z)^2} \text{ near } w = z.$$

It is symmetric in w and z . We will call $B(w, z)$ the *Bergmann Kernel*, following [3]. It is called the fundamental normalised differential of the second kind on C in [4]. Recall that a differential is *normalised* if its A -periods vanish and it is of the *second kind* if its residues vanish. It is used to express a normalised differential of the second kind in terms of local information around its poles.

For $2g - 2 + n > 0$, the poles of $\omega_n^g(p_1, \dots, p_n)$ occur at the branch points of x , and they are of order $6g - 4 + 2n$. Since each branch point α of x is simple, for any point $p \in C$ close to α there is a unique point $\hat{p} \neq p$ close to α such that $x(\hat{p}) = x(p)$. The recursive definition of $\omega_n^g(p_1, \dots, p_n)$ uses only local information around branch points of x and makes use of the well-defined map $p \mapsto \hat{p}$ there. The invariants are defined as follows.

$$(11) \quad \begin{aligned} \omega_1^0 &= ydx \\ \omega_2^0 &= B(w, z) \end{aligned}$$

For $2g - 2 + n > 0$,

$$(12) \quad \omega_{n+1}^g(z_0, z_S) = \sum_{\alpha} \operatorname{Res}_{z=\alpha} K(z_0, z) \left[\omega_{n+2}^{g-1}(z, \hat{z}, z_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J=S}} \omega_{|I|+1}^{g_1}(z, z_I) \omega_{|J|+1}^{g_2}(\hat{z}, z_J) \right]$$

where the sum is over branch points α of x , $S = \{1, \dots, n\}$, I and J are non-empty and

$$K(z_0, z) = \frac{-\int_{\hat{z}}^z B(z_0, z')}{2(y(z) - y(\hat{z}))dx(z)}$$

is well-defined in the vicinity of each branch point of x . Note that the quotient of a differential by the differential $dx(z)$ is a meromorphic function. The recursion (12) depends only on the meromorphic differential ydx and the map $p \mapsto \hat{p}$ around branch points of x .

2.1. Rational curves with two branch points. The simplest example of a plane curve with non-trivial Eynard-Orantin invariants is the rational curve $y^2 = x$ where the meromorphic function x defines a two-to-one branched cover with a single branch point. It is known as the Airy curve since the Eynard-Orantin invariants reproduce Kontsevich's generating function [6] for intersection numbers on the moduli space. In this paper we study rational curves such that x defines a two-to-one branched cover with two branch points.

Lemma 2.1. *If x is a two-to-one rational map on \mathbb{P}^1 with two branch points then we can parametrise the domain so that $x = a + b(z + 1/z)$.*

Proof. Using a conformal map we can arrange that the two branch points of x are $z = \pm 1$. The conformal map $z \mapsto (z + \lambda)/(\lambda z + 1)$ which fixes $z = \pm 1$ can be used to further arrange that $x(\infty) = \infty$. Since $x(z) - x(1)$ has a double root at $z = 1$ we have

$$x(z) - x(1) = \frac{(z-1)^2}{q_2 z^2 + q_1 z + q_0}, \quad x(\infty) = \infty \Rightarrow q_2 = 0, \quad x'(-1) = 0 \Rightarrow q_0 = 0$$

so put $a = x(1) - 2/q_1$ and $b = 1/q_1$, and the result follows. \square

Thus $x(z) = a + b(z + 1/z)$ and $y(z)$ is any analytic function defined on a domain in \mathbb{C} containing $z = \pm 1$ and satisfying $y'(\pm 1) \neq 0$. When y is not polynomial, for example y is rational or transcendental, we expand it as a series of polynomials in the following non-standard way. Given such $y(z)$, define the partial sums

$$y^{(N)}(z) = \sum_{k=0}^N (a_k + z b_k) (1 - z^2)^k$$

to agree with $y(z)$ at $z = \pm 1$ up to the N th derivatives. One can achieve this by expressing $y(z) = y_+(z) + y_-(z)$ where $y_{\pm}(z) = 1/2(y(z) \pm y(-z))$ and define $y_+^{(N)}(z) = \sum_{k=0}^N a_k (1 - z^2)^k$ where a_k are determined by the property

$$\frac{d^k y_+}{dz^k}(\pm 1) = \frac{d^k y_+^{(N)}}{dz^k}(\pm 1), \quad k = 0, \dots, N.$$

Similarly define b_k from $y_-(z)/z$.

The partial sums $\{y^{(N)}(z)\}$ do not necessarily converge to $y(z)$. For example,

$$y(z) = \ln z \sim \sum \frac{(1 - z^2)^k}{-2k}$$

is a divergent asymptotic expansion for $\ln(z)$ at $z = 0$ in the region $\text{Re}(z^2) > 0$.

The partial sums $y^{(N)}(z)$ are used in the recursions defining ω_n^g in place of $y(z)$ since they contain the same local information around $z = \pm 1$ up to order N . More precisely, to define ω_n^g for the curve $(x(z), y(z))$ it is sufficient to use $(x(z), y^{(N)}(z))$ for any $N \geq 6g - 6 + 2n$.

There are various ways to express a transcendental function as a limit of rational functions. The main benefit of the approach used here is the fact that the expressions which appear in the string and dilaton equations $x(z)^m y(z) \omega_n^g$ have poles only

at $z = \pm 1$ and 0 and ∞ , allowing one to translate properties of ω_n^g near $z = \pm 1$ to properties of ω_n^g near $z = \infty$, which is encoded by N_n^g .

Theorem 4 involves the expression $y(\mathcal{D})$ where \mathcal{D} is defined in the introduction, and in particular $I - \mathcal{D}$ is a discrete derivative. It is an easy fact (proved by induction) that for any degree d polynomial $p(n)$, high enough discrete derivatives vanish: $(1 - \mathcal{D})^k p(n) \equiv 0$ for $k > d$. Similarly, $1 - \mathcal{D}^2$ is a discrete derivative, and for a parity dependent quasi-polynomial $p(n)$ (so $p(n) = p_+(n)$ for n even and $p(n) = p_-(n)$ for n odd, where $p_{\pm}(n)$ are polynomials)

$$(1 - \mathcal{D}^2)^k p(n) \equiv 0, \quad k \text{ sufficiently large}$$

in fact $k > \text{maximum degree of } p_{\pm}(n)$. To make sense of (7) - (10), one must replace $y(\mathcal{D})$ with $y^{(N)}(D)$ for large enough N so that the left hand sides of (7) - (10) have only finitely many terms. This procedure is well-defined, since the vanishing of discrete derivatives ensures that the left hand sides of (7) - (10) are independent of the choice of N when it is large enough.

3. PROOFS

Proof of Theorem 1. Theorem 1 reflects three main properties of the multilinear differential $\omega_n^g(z_1, \dots, z_n)$ proven in [3]—it is meromorphic, with poles at $z_i = \pm 1$ of order $6g - 4 + 2n =: 2d + 2$ and residue 0, and possesses symmetry under $z_i \mapsto 1/z_i$.

Since all residues of ω_n^g vanish, the integral

$$\mathcal{F}_n^g(z_1, \dots, z_n) = \int_0^{z_1} \dots \int_0^{z_n} \omega_n^g(z'_1, \dots, z'_n)$$

is a well-defined meromorphic function that vanishes when any $z_i = 0$ and has poles of order $2d + 1$ at $z_i = \pm 1$. Write this rational function as

$$\mathcal{F}_n^g(z_1, \dots, z_n) = \frac{\sum_{0 \leq k_i \leq 4d+2} p_{\mathbf{k}} z_1^{k_1} \dots z_n^{k_n}}{\prod_{i=1}^n (1 - z_i^2)^{2d+1}}$$

where the $p_{\mathbf{k}} = p_{k_1, \dots, k_n} \in \mathbb{C}$ and the degree of the numerator is small enough to avoid a pole at infinity.

The Taylor expansion

$$\frac{1}{(1 - z^2)^{2d+1}} = \frac{1}{(2d)!} \frac{d^{2d}}{d(z^2)^{2d}} \sum_{m=0}^{\infty} z^{2m} = \sum_{m=0}^{\infty} \binom{m+2d}{2d} z^{2m}$$

has quasi-polynomial coefficients, meaning that the coefficients of z^b are described by two polynomials in b —when b is odd the coefficient of z^b is the zero polynomial and when b is even the coefficient of z^b is a degree $2d$ polynomial in b . More generally, the Taylor expansion of $\mathcal{F}_n^g(z_1, \dots, z_n)$ about $z_i = 0$ has quasi-polynomial coefficients, depending on parity. When $n = 1$,

$$\frac{\sum p_k z^k}{(1 - z^2)^{2d+1}} = \sum_{k,m} p_k \binom{m+2d}{2d} z^{2m+k} = \sum_{b \geq 0} N_1^g(b) z^b.$$

The coefficient of z^b consists of all terms where $2m + k = b$, hence the odd part of $p(z) = \sum p_k z^k$ gives rise to a degree $2d$ polynomial representing $N_1^g(b)$ when b

is odd, and the even part of $p(z)$ gives rise to a degree $2d$ polynomial representing $N_1^g(b)$ when b is even. Similarly,

$$\begin{aligned} \mathcal{F}_n^g(z_1, \dots, z_n) &= \sum_{k_1, \dots, k_n=0}^{4d+2} p_{\mathbf{k}} z_1^{k_1} \dots z_n^{k_n} \prod_{i=1}^n \sum_{m_i=0}^{\infty} \binom{m_i + 2d}{2d} z_i^{2m_i} \\ &= \sum_{k_1, \dots, k_n=0}^{4d+2} \sum_{m_i \geq 0} p_{\mathbf{k}} \prod_{i=1}^n \binom{m_i + 2d}{2d} z_i^{2m_i + k_i} \\ &=: \sum_{b_i > 0} N_n^g(b_1, \dots, b_n) z_1^{b_1} \dots z_n^{b_n} \end{aligned}$$

expresses $N_n^g(b_1, \dots, b_n)$ as the sum over the terms with $2m_i + k_i = b_i$ which is a quasi-polynomial depending on the parity of the b_i . By symmetry of the z_i , N_n^g does not depend on which b_i are odd but only how many. Hence we write

$$N_n^g(b_1, \dots, b_n) = N_{n,k}^g(b_1, \dots, b_n), \quad k = \text{number of odd } b_i.$$

Each binomial coefficient and hence each polynomial $N_{n,k}^g(b_1, \dots, b_n)$ is a polynomial of degree $2d$ in each b_i . The stronger fact that they have *homogeneous* degree $2d$ in the b_i is a consequence of Theorem 2.

It remains to show that N_n^g is a quasi-polynomial in the b_i^2 . Equivalently, we will show that $b_1 \dots b_n N_n^g(b_1, \dots, b_n)$ is odd in each b_i using symmetries of

$$\omega_n^g = \sum_{b_i > 0} b_1 \dots b_n N_n^g(b_1, \dots, b_n) z_1^{b_1-1} \dots z_n^{b_n-1} dz_1 \dots dz_n.$$

Lemma 3.1. *A meromorphic 1-form on \mathbb{P}^1 with poles at $z = \pm 1$ has the following related expansions around $z = 0$*

$$(13) \quad \omega(z) = \sum_{n=1}^{\infty} p(n) z^{n-1} dz \quad \Leftrightarrow \quad \omega\left(\frac{1}{z}\right) = \sum_{n=1}^{\infty} p(-n) z^{n-1} dz$$

where $p(n)$ is a quasi-polynomial depending on the parity of n .

Proof. We can express ω as a rational function with numerator a polynomial of degree small enough so that there are no poles at infinity. In particular, by linearity it is enough to prove the lemma when the numerator is a monomial so

$$\omega(z) = \frac{z^k dz}{(1-z^2)^{m+1}} \quad \text{and} \quad \omega\left(\frac{1}{z}\right) = (-1)^m \frac{z^{2m-k} dz}{(1-z^2)^{m+1}}.$$

From the expansion

$$\frac{1}{(1-z^2)^{m+1}} = \sum \binom{n+m}{m} z^{2n}$$

one gets

$$\frac{z^k dz}{(1-z^2)^{m+1}} = \sum \binom{n+m}{m} z^{k+2n} dz = \sum p_k(b) z^{b-1} dz$$

where

$$p_k(b) = \begin{cases} 0, & b \equiv k \pmod{2} \\ \binom{(b-k-1)/2+m}{m}, & b \not\equiv k \pmod{2} \end{cases}.$$

Also,

$$p_{2m-k}(b) = \binom{(b-2m+k-1)/2+m}{m} = \binom{(b+k-1)/2}{m} = (-1)^m p_k(-b)$$

(for $b \not\equiv k \pmod{2}$ and $p_{2m-k}(b) = 0$ for $b \equiv k \pmod{2}$ so the above equation holds for all b .) Hence (13) holds when ω has a monomial numerator and hence for all rational ω and the lemma is proven. \square

An immediate corollary of the lemma is that if $\omega(z) = \omega(1/z)$ then the quasi-polynomial $p(n)$ is even in n and if $\omega(z) = -\omega(1/z)$ then $p(n)$ is odd in n . A consequence of a more general result in [3] is the symmetry

$$\omega_n^g(z_1, \dots, z_n) = -\omega_n^g(1/z_1, \dots, z_n)$$

and similarly for each variable z_i . Hence $b_1 \dots b_n N_n^g(b_1, \dots, b_n)$ is an odd quasi-polynomial in each b_i as required. \square

Remark. The expansion (2) of $\omega_n^g(z_1, \dots, z_n)$ around $(z_1, \dots, z_n) = (0, \dots, 0)$ defines $N_n^g(b_1, \dots, b_n)$ only for $(b_1, \dots, b_n) \in \mathbb{Z}_+^n$. One can make sense of $b_i = 0$ using the polynomial representation $N_{n,k}^g(b_1, \dots, b_n)$ of $N_n^g(b_1, \dots, b_n)$ for $k =$ number of odd b_i . In terms of ω_n^g one has the following

$$N_1^g(0) = \int_{-\infty}^0 \omega_1^g(z)$$

and more generally, $b_1 \dots b_k N_n^g(b_1, \dots, b_k, 0, \dots, 0)$ is the coefficient of $z_1^{b_1-1} \dots, z_k^{b_k-1}$ in the expansion around $(z_1, \dots, z_k) = (0, \dots, 0)$ of $\int_{z_{k+1}=\infty}^0 \dots \int_{z_n=\infty}^0 \omega_n^g(z_1, \dots, z_n)$.

Proof of Theorem 2. The proof uses the behaviour of ω_n^g near the branch points $z_i = \pm 1$. Express ω_n^g as a rational function

$$\omega_n^g = \frac{\sum_{k_1, \dots, k_n=0}^{4d+2} c_{\mathbf{k}} z_1^{k_1} \dots z_n^{k_n}}{\prod_{i=1}^n (1 - z_i^2)^{2d+2}} dz_1 \dots dz_n$$

for $c_{\mathbf{k}} \in \mathbb{C}$, and $d = 3g - 3 + n$. Consider the change of variables $z_i = \epsilon_i + sx_i$ where $\epsilon_i \in \{\pm 1\}$, $s \in \mathbb{R}$ is small and x_i is a local coordinate on the spectral curve. The asymptotic behaviour of ω_n^g near $z_i = \pm 1$ corresponds to $s \rightarrow 0$ for all combinations of the ϵ_i . This change gives:

$$\omega_n^g = \frac{\sum_{k_1, \dots, k_n=0}^{4d+2} c_{\mathbf{k}} (\epsilon_1 + sx_1)^{k_1} \dots (\epsilon_n + sx_n)^{k_n}}{s^{(2d+2)n} \prod_{i=1}^n x_i^{2d+2} (2\epsilon_i + sx_i)^{2d+2}} s^n \prod_{i=1}^n dx_i,$$

and we must find a minimal $q = q(\epsilon_i) \in \{0, 1, \dots, 4d+2\}$ so that the coefficient of s^q in the numerator is the first non vanishing. For example, if $q > 0$ this tells us

$$\sum_{k_1, \dots, k_n=0}^{4d+2} c_{\mathbf{k}} \prod_{i=1}^n \epsilon_i^{k_i} = 0,$$

and if $q > 1$, then the coefficient of $s^1 = 0$. That is:

$$\sum_{k_1, \dots, k_n=0}^{4d+2} c_{\mathbf{k}} \prod_{i=1}^n \epsilon_i^{k_i} \left(\binom{k_1}{1} \frac{x_1}{\epsilon_1} + \dots + \binom{k_n}{1} \frac{x_n}{\epsilon_n} \right) = 0$$

and by equating coefficients of x_j , for $1 \leq j \leq n$:

$$\sum_{k_1, \dots, k_n=0}^{4d+2} c_{\mathbf{k}} \prod_{i=1}^n \epsilon_i^{k_i} \frac{k_j}{\epsilon_j} = 0.$$

For a general q and $\alpha = (\alpha_1, \dots, \alpha_n)$

$$\sum_{k_1, \dots, k_n=0}^{4d+2} c_{\mathbf{k}} \binom{k_1}{\alpha_1} \dots \binom{k_n}{\alpha_n} \epsilon_1^{k_1-\alpha_1} \dots \epsilon_n^{k_n-\alpha_n} x_1^{\alpha_1} \dots x_n^{\alpha_n} = 0 \quad \text{if } |\alpha| < q.$$

Thus inductively one gets

$$(14) \quad \sum_{k_1, \dots, k_n=0}^{4d+2} c_{\mathbf{k}} \epsilon_1^{k_1-\alpha_1} \dots \epsilon_n^{k_n-\alpha_n} k_1^{\alpha_1} \dots k_n^{\alpha_n} = 0 \quad \text{if } |\alpha| < q.$$

This means that the dominant asymptotic term as $s \rightarrow 0$ will look like:

$$\omega_n^g \sim \frac{1}{s^{(2d+1)n-q} \prod_{i=1}^n x_i^{2d+2}} \sum_{k_1, \dots, k_n=0}^{4d+2} \sum_{|\alpha|=q} \frac{c_{\mathbf{k}}}{2^{(2d+2)n} \alpha!} \prod_{i=1}^n \epsilon_i^{k_i-\alpha_i} k_i^{\alpha_i} x_i^{\alpha_i} dx_i.$$

In [2] it is shown that as z_1, \dots, z_n tends to the branch points $\epsilon_1, \dots, \epsilon_n$, $\epsilon_j = \pm 1$

$$\omega_n^g \sim \begin{cases} s^{6-6g-3n} [\frac{1}{2} x''(\epsilon_i) y'(\epsilon_i)]^{2-2g-n} \omega_n^g[Airy], & \text{for all } \epsilon_i \text{ the same} \\ \text{lower order asymptotics,} & \text{for mixed } \epsilon_i \end{cases}$$

where the Airy curve is given by $y^2 = x$. Thus $q = 2d(n-1)$ if all ϵ_i are the same and $q > 2d(n-1)$ for all other combinations.

From [3] there is a relationship between $\omega_n^g[Airy]$ and intersection numbers of tautological line bundles over the moduli space.

$$(15) \quad \omega_n^g[Airy](z_S) = \frac{x''(0)^{2-2g-n}}{2^{3g-3+n}} \sum_{|\beta|=d} \prod_{i=1}^n \frac{(2\beta_i+1)!}{\beta_i!} \frac{dz_i}{z_i^{2\beta_i+2}} \langle \tau_{\beta_1} \dots \tau_{\beta_n} \rangle,$$

where we have used Witten's [11] notation $\langle \tau_{\beta_1} \dots \tau_{\beta_n} \rangle = \int_{\overline{\mathcal{M}}_{g,n}} c_1(L_1)^{\beta_1} \dots c_1(L_n)^{\beta_n}$ and $x''(0) = 2$. From this we discover that if all $\epsilon_i = 1$:

$$\begin{aligned} & \frac{1}{\prod_{i=1}^n x_i^{2d+2}} \sum_{k_1, \dots, k_n=0}^{4d+2} \sum_{|\alpha|=2d(n-1)} \frac{c_{\mathbf{k}}}{2^{(2d+2)n} \alpha!} \prod_{i=1}^n k_i^{\alpha_i} x_i^{\alpha_i} dx_i \\ &= \sum_{k_1, \dots, k_n=0}^{4d+2} \sum_{|\alpha|=2d(n-1)} \frac{c_{\mathbf{k}}}{2^{(2d+2)n}} \prod_{i=1}^n \frac{k_i^{\alpha_i} x_i^{\alpha_i-2d-2}}{\alpha_i!} dx_i \\ &= \frac{\{x''(1)y'(1)\}^{2-2g-n}}{2^{3g-3+n}} \sum_{|\beta|=d} \prod_{i=1}^n \frac{(2\beta_i+1)!}{\beta_i!} \frac{dx_i}{x_i^{2\beta_i+2}} \langle \tau_{\beta_1} \dots \tau_{\beta_n} \rangle. \end{aligned}$$

and so each α_i must be even and $0 \leq \alpha_i \leq 2d$. By equating powers of x_i (that is, extracting the partition where $\alpha_i = 2d - 2\beta_i$) one gets the relation

$$(16) \quad \sum_{k_1, \dots, k_n=0}^{4d+2} \frac{c_{\mathbf{k}}}{2^{(2d+2)n}} \prod_{i=1}^n \frac{k_i^{2d-2\beta_i}}{(2d-2\beta_i)!} = \frac{\{x''(1)y'(1)\}^{2-2g-n}}{2^{3g-3+n}} \prod_{i=1}^n \frac{(2\beta_i+1)!}{\beta_i!} \langle \tau_{\beta_1} \dots \tau_{\beta_n} \rangle.$$

Similarly, when all $\epsilon_i = -1$ one gets

$$(17) \quad \sum_{k_1, \dots, k_n=0}^{4d+2} \frac{c_{\mathbf{k}}}{2^{(2d+2)n}} \prod_{i=1}^n \frac{(-1)^{k_i} k_i^{2d-2\beta_i}}{(2d-2\beta_i)!} = \frac{\{x''(-1)y'(-1)\}^{2-2g-n}}{2^{3g-3+n}} \prod_{i=1}^n \frac{(2\beta_i+1)!}{\beta_i!} \langle \tau_{\beta_1} \dots \tau_{\beta_n} \rangle.$$

and when there is a mix of ϵ_i 's, q will be greater, introducing more vanishing:

$$(18) \quad \sum_{k_1, \dots, k_n=0}^{4d+2} \frac{c_{\mathbf{k}}}{2^{(2d+2)n}} \prod_{i=1}^n \frac{\epsilon_i k_i^{2d-2\beta_i}}{(2d-2\beta_i)!} = 0 \quad \text{for } |\beta| = d.$$

Equations (16), (17) and (18) translate to an analogous equation (21) for coefficients of the polynomials $N_{n,k}^g$. To show this we now study the polynomials in terms of the coefficients $c_{\mathbf{k}}$. As in the proof of Theorem 1, the Taylor expansion of ω_n^g about $z_i = 0$ can be written:

$$\begin{aligned} \omega_n^g &= \sum_{k_1, \dots, k_n=0}^{4d+2} \sum_{b_1, \dots, b_n=0}^{\infty} c_{\mathbf{k}} \prod_{i=1}^n \binom{(b_i+1-k_i)/2+2d}{2d+1} z_i^{b_i-1} dz_i \\ &= \frac{2^{-(2d+1)n}}{(2d+1)!^n} \sum_{k_1, \dots, k_n=0}^{4d+2} \sum_{b_1, \dots, b_n=0}^{\infty} c_{\mathbf{k}} \prod_{i=1}^n \left(\sum_{j=0}^{2d+1} \sigma_j(k_i) b_i^{2d+1-j} \right) z_i^{b_i-1} dz_i \\ &= \sum_{b_i > 0}^{\infty} b_1 \dots b_n N_n^g(b_1, \dots, b_n) z_1^{b_1-1} \dots z_n^{b_n-1} dz_1 \dots dz_n \end{aligned}$$

where for b_i even, respectively odd, we only sum over the odd, respectively even, k_i and $\sigma_j(k_i)$ (= coefficient of b_i^{2d+1-j} in $(b_i+4d+1-k_i) \dots (b_i+3-k_i)(b_i+1-k_i)$) is a degree j polynomial in k_i .

Thus the homogeneous degree $2q$ terms of the quasi-polynomial N_n^g are

$$(19) \quad \left(\prod_{i=1}^n b_i \right) N_n^g[\text{degree } 2q] = \frac{2^{-(2d+1)n}}{(2d+1)!^n} \sum_{k_1, \dots, k_n=0}^{4d+2} \sum_{|\beta|=q} c_{\mathbf{k}} \prod_{i=1}^n \sigma_{2d-2\beta_i}(k_i) b_i^{2\beta_i+1}$$

where we are still summing over parity dependent \mathbf{k} .

The equations (14), (16), (17) and (18) give identities for sums over *all* k_i of $c_{\mathbf{k}}$ times monomials in k_i and we wish to apply these to (19) which consists of coefficients that sum over only some of the k_i , depending on parity. To remedy this, we add together the different polynomials representing N_n^g for every possible parity. This removes the restriction on the \mathbf{k} summand.

For $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$, define $v_{\beta}^{\{i_1, \dots, i_k\}}$ to be the coefficient of $b_1^{2\beta_1} \dots b_n^{2\beta_n}$ in $N_{n,k}^g$ with b_{i_1}, \dots, b_{i_k} odd. For $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \{\pm 1\}^n$ define $\epsilon^I = \prod_{k \in I} \epsilon_k$. Since $\sigma_j(k_i)$ is a polynomial of degree j in k_i , if $|\beta| = q > d$, then the homogeneous degree in k_1, k_2, \dots, k_n of the product of $\sigma_{2d-2\beta_i}(k_i)$ is small enough that (14) implies the vanishing of each of the coefficients. In other words we have shown that

$$(20) \quad \sum_{I \subset \{1, \dots, n\}} \epsilon^I v_{\beta}^I = 0, \quad |\beta| > d.$$

If $|\beta| = q = d$, then the only non zero sums resulting from the $\sigma_{2d-2\beta_i}(k_i)$ are the top powers of the k_i , that is $\prod_{i=1}^n (-k_i)^{2d-2\beta_i}$, since any component with a smaller

power of k_i 's will vanish when summed again by (14). There will be $\prod_{i=1}^n \binom{2d+1}{2d-2\beta_i}$ of these. This leaves

$$\begin{aligned} & \sum_{\text{parities}} N_n^g[\text{degree } 2d] \\ &= \frac{2^{-(2d+1)n}}{(2d+1)!^n} \sum_{k_1, \dots, k_n=0}^{4d+2} \sum_{|\beta|=d} c_{\mathbf{k}} \prod_{i=1}^n \binom{2d+1}{2d-2\beta_i} (-1)^{2d-2\beta_i} k_i^{2d-2\beta_i} b_i^{2\beta_i} \\ &= \frac{1}{2^{(2d+1)n}} \sum_{k_1, \dots, k_n=0}^{4d+2} \sum_{|\beta|=d} c_{\mathbf{k}} \prod_{i=1}^n \frac{k_i^{2d-2\beta_i}}{(2d-2\beta_i)!} \frac{b_i^{2\beta_i}}{(2\beta_i+1)!} \end{aligned}$$

thus

$$\sum_{I \subset \{1, \dots, n\}} v_{\beta}^I = \frac{1}{2^{(2d+1)n}} \sum_{k_1, \dots, k_n=0}^{4d+2} c_{\mathbf{k}} \prod_{i=1}^n \frac{k_i^{2d-2\beta_i}}{(2d-2\beta_i)!(2\beta_i+1)!}$$

where we are again extending to the full sum over all \mathbf{k} 's. More generally

$$\sum_{I \subset \{1, \dots, n\}} \epsilon^I v_{\beta}^I = \frac{1}{2^{(2d+1)n}} \sum_{k_1, \dots, k_n=0}^{4d+2} c_{\mathbf{k}} \prod_{i=1}^n \frac{\epsilon_i^{k_i+1} k_i^{2d-2\beta_i}}{(2d-2\beta_i)!(2\beta_i+1)!}$$

Now we can see that these are (up to simple combinatorial factors) the asymptotic formulas (16), (17) and (18)! We now have expressions for these in terms of intersection numbers.

(21)

$$\sum_{I \subset \{1, \dots, n\}} \epsilon^I v_{\beta}^I = \begin{cases} \frac{\{x''(1)y'(1)\}^{2-2g-n}}{2^{3g-3}\beta!} \langle \tau_{\beta_1} \dots \tau_{\beta_n} \rangle, & \epsilon_i = 1 \text{ for all } i \\ \frac{(-1)^n \{x''(-1)y'(-1)\}^{2-2g-n}}{2^{3g-3}\beta!} \langle \tau_{\beta_1} \dots \tau_{\beta_n} \rangle, & \epsilon_i = -1 \text{ for all } i, \quad |\beta| = d. \\ 0 & \text{otherwise.} \end{cases}$$

By varying $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \{\pm 1\}^n$, (20) and (21) are sets of 2^n equations with 2^n unknowns v_{β}^I that can be uniquely solved. When $n = 1$, the two equations use the matrix $J = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, and more generally the 2^n equations use the $2^n \times 2^n$ matrix given by the tensor product $M = J^{\otimes n}$ —equivalently M is the linear map induced by J on the tensor product $(\mathbb{C}^2)^{\otimes n}$. The matrix M is orthogonal (up to a 2^n scaling factor)

$$MM^T = (JJ^T)^{\otimes n} = (2I)^{\otimes n} = 2^n I.$$

Assemble v_{β}^I into a 2^n -vector v_{β} . Thus (20) becomes

$$(20') \quad Mv_{\beta} = 0, \quad |\beta| > d.$$

and (21) becomes $Mv_{\beta} =$ the right hand side of (21) or more explicitly

$$(21') \quad Mv_{\beta} = \begin{pmatrix} \{x''(1)y'(1)\}^{2-2g-n} \\ 0 \\ \vdots \\ (-1)^n \{x''(-1)y'(-1)\}^{2-2g-n} \end{pmatrix} \frac{\langle \tau_{\beta_1} \dots \tau_{\beta_n} \rangle}{2^{3g-3}\beta!}, \quad |\beta| = d$$

where the first and last rows of M are the 2^n -vectors $e_0 = (1, 1, \dots)$ and $e_1 = \{(-1)^I\}$ corresponding to $\epsilon_i = 1$ for all i , respectively $\epsilon_i = -1$ for all i . In particular, M

is invertible so equations (20') and (21') have the unique solutions $v_\beta = 0$ when $|\beta| > d$ and when $|\beta| = d$, v_β lies in the plane spanned by e_0 and e_1 . Explicitly

$$\begin{aligned} v_\beta &= \frac{\langle \tau_{\beta_1} \dots \tau_{\beta_n} \rangle}{2^{3g-3+n} \beta!} \left(\{x''(1)y'(1)\}^{2-2g-n} e_0 + (-1)^n \{x''(-1)y'(-1)\}^{2-2g-n} e_1 \right) \\ &= \frac{\langle \tau_{\beta_1} \dots \tau_{\beta_n} \rangle}{x''(1)^{2g-2+n} 2^{3g-3+n} \beta!} (y'(1)^{2-2g-n} e_0 + y'(-1)^{2-2g-n} e_1) \end{aligned}$$

where we have used $x''(-1) = -x''(1)$. Equivalently

$$v_\beta^I = \frac{y'(1)^{2-2g-n} + (-1)^{|I|} y'(-1)^{2-2g-n}}{x''(1)^{2g-2+n} 2^{3g-3+n} \beta!} \langle \tau_{\beta_1} \dots \tau_{\beta_n} \rangle.$$

In particular, the maximal homogeneous degree of each polynomial $N_{n,k}^g(b_1, \dots, b_n)$ is $2d = 6g - 6 + 2n$ and the top coefficients are given in terms of intersection numbers as claimed. \square

4. STRING AND DILATON EQUATIONS

The Eynard-Orantin invariants ω_n^g satisfy the following *string equations* [3].

$$(22) \quad \sum_{\alpha} \operatorname{Res}_{z=\alpha} x^m y \omega_{n+1}^g(z, z_S) = - \sum_{j=1}^n dz_j \frac{\partial}{\partial z_j} \left(\frac{x^m(z_j) \omega_n^g(z_S)}{dx(z_j)} \right)$$

for $m = 0, 1$ or 2 , α the poles of ydx and $z_S = \{z_1, \dots, z_n\}$. Note that the $m = 2$ case only works for y being a sum of monomials, since the proof of the above equation in [3] requires $\frac{x^m(z)}{dx(z)}$ to not have a pole at any of the poles of $ydx(z)$.

4.1. Proof of Theorem 4. If $y(z)$ is a polynomial, represent it as a finite sum $y(z) = y_0 + y_1 z + y_2 z^2 + \dots$. More generally, as described in Section 2, we can approximate any analytic $y(z)$ by a polynomial $y^{(N)}(z)$ which agrees with $y(z)$ at $z = \pm 1$ up to the N th derivatives. Express the polynomial as a finite sum $y^{(N)}(z) = y_0 + y_1 z + y_2 z^2 + \dots$. Note that when $y(z)$ is not a polynomial then the finite sum $y_0 + y_1 z + y_2 z^2 + \dots$ is not a partial sum for a Taylor series of $y(z)$ at $z = 0$. In the following, we choose $N \geq 6g - 6 + 2(n+1)$ so that $y(z)$ can be replaced in the left hand side of (22) by $y^{(N)}(z)$ in the residue calculations.

$$\boxed{m = 0}$$

$$\begin{aligned} \sum_{\alpha=\pm 1} \operatorname{Res}_{z=\alpha} y(z) \omega_{n+1}^g(z, z_S) &= - \sum_{\alpha=0, \infty} \operatorname{Res}_{z=\alpha} (y_1 z + y_2 z^2 + \dots) \omega_{n+1}^g(z, z_S) \\ &= - \operatorname{Res}_{z=\infty} (y_1 z + y_2 z^2 + \dots) \omega_{n+1}^g(z, z_S) \\ &= - \operatorname{Res}_{z=0} \left(\frac{y_1}{z} + \frac{y_2}{z^2} + \dots \right) (-\omega_{n+1}^g(z, z_S)) \end{aligned}$$

The coefficient of $\prod_{i=1}^n b_i z_i^{b_i-1}$ in the expansion about zero is

$$y_1 N_{n+1}^g(b_1, \dots, b_n, 1) + 2y_2 N_{n+1}^g(b_1, \dots, b_n, 2) + \dots = \sum_{k=1}^{\infty} k y_k N_{n+1}^g(b_1, \dots, b_n, k)$$

which can be expressed as $\mathcal{D}y(\mathcal{D}) \{m N_{n+1}^g(m, b_S)\}_{m=-1}$ as required.

The right hand side of (22), as shown in [8], expands as

$$(23) \quad \sum_{j=1}^n \frac{\partial}{\partial z_j} [(z_j^2 + z_j^4 + z_j^6 + \dots) \omega_n^g(z_S)]$$

and the coefficient of $\prod_{i=1}^n b_i z_i^{b_i-1}$ in the expansion about zero is given by

$$\sum_{j=1}^n \sum_{\substack{k=1 \\ k \not\equiv b_j(2)}}^{b_j} k N_n^g(b_1, \dots, b_n)|_{b_j=k}$$

$$\boxed{m=1}$$

$$\begin{aligned} & \sum_{\alpha=\pm 1} \operatorname{Res}_{z=\alpha} x(z) y(z) \omega_{n+1}^g(z, z_S) \\ &= - \sum_{\alpha=0, \infty} \operatorname{Res}_{z=\alpha} (y_1 + y_2 z + (y_1 + y_3) z^2 + (y_2 + y_4) z^3 + \dots) \omega_{n+1}^g(z, z_S) \\ &= - \operatorname{Res}_{z=\infty} (y_1 + y_2 z + (y_1 + y_3) z^2 + (y_2 + y_4) z^3 + \dots) \omega_{n+1}^g(z, z_S) \\ &= \operatorname{Res}_{z=0} (y_1 + \frac{y_2}{z} + \frac{y_1 + y_3}{z^2} + \frac{y_2 + y_4}{z^3} + \dots) \omega_{n+1}^g(z, z_S). \end{aligned}$$

The coefficient of $\prod_{i=1}^n b_i z_i^{b_i-1}$ in the expansion about zero is

$$y_2 N_{n+1}^g(1, b_1, \dots, b_n) + \sum_{k=2}^{\infty} k (y_{k-1} + y_{k+1}) N_{n+1}^g(k, b_1, \dots, b_n)$$

and if we add $-y_0 N_{n+1}^g(-1, b_1, \dots, b_n) + y_0 N_{n+1}^g(1, b_1, \dots, b_n) = 0$ this can be expressed as $(1 + \mathcal{D}^2) y(\mathcal{D}) \{m N_{n+1}^g(m, b_S)\}_{m=-1}$.

The right hand side of (22) can be expanded as

$$(24) \quad \sum_{j=1}^n \frac{\partial}{\partial z_j} [(z_j + 2z_j^3 + 2z_j^5 + \dots) \omega_n^g(z_S)]$$

and the coefficient of $\prod_{i=1}^n b_i z_i^{b_i-1}$ in the expansion about zero is

$$2 \sum_{j=1}^n \sum_{\substack{k=1 \\ k \equiv b_j(2)}}^{b_j} k N_n^g(b_1, \dots, b_n)|_{b_j=k} + b_j N_n^g(b_1, \dots, b_n).$$

$$\boxed{m = 2}$$

$$\begin{aligned}
& \sum_{\alpha=\pm 1} \operatorname{Res}_{z=\alpha} x^2(z)y(z)\omega_{n+1}^g(z, z_S) \\
&= - \sum_{\alpha=0, \infty} \operatorname{Res}_{z=\alpha} \left(\frac{y_1}{z} + y_2 + (y_3 + 2y_1)z + \dots \right) \omega_{n+1}^g(z, z_S) \\
&= - \operatorname{Res}_{z=0} \frac{y_1}{z} \omega_{n+1}^g(z, z_S) - \operatorname{Res}_{z=\infty} (y_2 + (y_3 + 2y_1)z + \dots) \omega_{n+1}^g(z, z_S) \\
&= \operatorname{Res}_{z=0} \left(-\frac{y_1}{z} + \frac{y_3 + 2y_1}{z} + \frac{y_4 + 2y_2}{z^2} + \frac{y_5 + 2y_3 + y_1}{z^3} + \dots \right) \omega_{n+1}^g(z, z_S)
\end{aligned}$$

and the right hand side expands as:

$$(25) \quad \sum_{j=1}^n \frac{\partial}{\partial z_j} [(1 + 3z_j^2 + 4z_j^4 + 4z_j^6 + \dots) \omega_n^g(z_S)].$$

In this case it is convenient to subtract four times the $m = 0$ case (equivalently we put $x^2 - 4$ in place of x^2 in (22).) Once again, collecting the coefficient of $\prod_{i=1}^n b_i z_i^{b_i-1}$ in the expansion about zero of both sides gives the result.

4.2. Dilaton. The Eynard-Orantin invariants also satisfy the *dilaton equation*.

$$(26) \quad \sum_{\alpha=\pm 1} \operatorname{Res}_{z=\alpha} \Phi(z) \omega_{n+1}^g(z, z_S) = (2g - 2 + n) \omega_n^g(z_S)$$

where $d\Phi = ydx$. The function Φ is well-defined up to a constant in a neighbourhood of each branch point and the left hand side of (26) is independent of the choice of constant.

Proof of equation (10). Starting from the Dilaton equation proven in [3], we let $\Phi(z)$ be an arbitrary anti derivative of ydx . That is, $d\Phi(z) = ydx(z)$. Then

$$(27) \quad \sum_{\alpha=\pm 1} \operatorname{Res}_{z=\alpha} \Phi(z_{n+1}) \omega_{n+1}^g(z, z_S) = (2g - 2 + n) \omega_n^g(z_S)$$

We can manipulate this, using integration by parts to rewrite the left hand side as:

$$\begin{aligned}
\sum_{\alpha=\pm 1} \operatorname{Res}_{z=\alpha} \Phi(z_{n+1}) \omega_{n+1}^g(z, z_S) &= - \sum_{\alpha=\pm 1} \operatorname{Res}_{z=\alpha} d\Phi(z) \int_0^z \omega_{n+1}^g(z', z_S) \\
&= - \sum_{\alpha=\pm 1} \operatorname{Res}_{z=\alpha} \left(\frac{-y_1}{z} - y_2 + (y_1 - y_3)z + (y_2 - y_4)z^2 + \dots \right) dz \int_0^z \omega_{n+1}^g(z', z_S) \\
&= \operatorname{Res}_{z=\infty} \left(\frac{-y_1}{z} - y_2 + (y_1 - y_3)z + (y_2 - y_4)z^2 + \dots \right) dz \int_0^z \omega_{n+1}^g(z', z_S) \\
&= - \operatorname{Res}_{z=\infty} \left(-y_2 z + \frac{y_1 - y_3}{2} z^2 + \dots \right) \omega_{n+1}^g(z, z_S) - \operatorname{Res}_{z=\infty} \frac{y_1 dz}{z} \int_0^z \omega_{n+1}^g(z', z_S) \\
&= \operatorname{Res}_{z=0} \left(-\frac{y_2}{z} + \frac{y_1 - y_3}{2z^2} + \dots \right) \omega_{n+1}^g(z, z_S) - \operatorname{Res}_{z=\infty} \frac{y_1}{z} \int_0^z \omega_{n+1}^g(z', z_S)
\end{aligned}$$

where lines two to three uses the sum of residues of a meromorphic function is zero, and lines three to four uses integration by parts. The last residue about the simple pole $z = \infty$ can be computed, since $\int_0^z \omega_{n+1}^g(z', z_S)$ is analytic there. Thus

$$\operatorname{Res}_{z=\infty} \frac{y_1 dz}{z} \int_0^z \omega_{n+1}^g(z', z_S) = -y_1 \int_0^\infty \omega_{n+1}^g(z', z_S)$$

and as in the remark after the proof of Theorem 1 we can extract the coefficient of $\prod_{i=1}^n b_i z_i^{b_i-1}$ in the expansion about zero to get $y_1 N_{n+1}^g(0, b_1, \dots, b_n)$. Hence the total coefficient of $\prod_{i=1}^n b_i z_i^{b_i-1}$ in the expansion about zero is

$$-y_1 N_{n+1}^g(0, b_1, \dots, b_n) - y_2 N_{n+1}^g(1, b_1, \dots, b_n) + \sum_{k=2}^{\infty} (y_{k-1} - y_{k+1}) N_{n+1}^g(k, b_1, \dots, b_n)$$

and if we add $y_0 N_{n+1}^g(1, b_1, \dots, b_n) - y_0 N_{n+1}^g(-1, b_1, \dots, b_n) = 0$ this can be expressed as $-(1 - \mathcal{D}^2)y(\mathcal{D}) \{N_{n+1}^g(m, b_S)\}_{m=-1}$ as required. \square

Remark. It was necessary in the proof of Theorem 4 that the $b_i > 0$. The equations immediately extend to allow all b_i , since the left hand side and right hand side are polynomials that agree at infinitely many values in each variable hence they coincide. For example, when $x = z + 1/z$ and $y = z$ the dilaton equation yields

$$N_{n+1,k}^g(2, b_1, \dots, b_n) - N_{n+1,k}^g(0, b_1, \dots, b_n) = (2g - 2 + n) N_{n,k}^g(b_1, \dots, b_n).$$

If $b_j = 0$ in the string equation then the sum on the right hand side corresponding to j is empty as in the following $n = 0$ case.

4.3. **$n = 0$ case.** We can set $n = 0$ in the string equations and the dilaton equation to get interesting results. The string equations give vanishing results for N_1^g which are useful to help calculate N_1^g and in practice to check calculations.

Proposition 4.1. *For $g > 1$, the recursions (3, 4, 5) still hold when $n = 0$ and the right hand side is set to zero, leading to vanishing results.*

Proof. This comes directly from the proof of the string equation in [3], applied to the case of $n = 0$. For α the poles of $yx(z)$, $m = 0, 1$ or 2 and P_k^g defined by theorem 4.5 in [2] we have

$$\begin{aligned}
& \sum_{\alpha} \operatorname{Res}_{z=\alpha} x(z)^m y(z) \omega_1^g(z) \\
&= -\frac{1}{2} \sum_{\alpha} \operatorname{Res}_{z=\alpha} \frac{x(z)^m}{dx(z)} (-2y(z)dx(z)\omega_1^g(z) + \sum_{h=0}^g \omega_1^h(z)\omega_1^{g-h}(z) + \omega_2^{g-1}(z, z)) \\
&= \frac{1}{2} \sum_{a=\pm 1} \operatorname{Res}_{z=a} \frac{x(z)^m}{dx(z)} (-2y(z)dx(z)\omega_1^g(z) + \sum_{h=0}^g \omega_1^h(z)\omega_1^{g-h}(z) + \omega_2^{g-1}(z, z)) \\
&= \frac{1}{4} \sum_{a=\pm 1} \operatorname{Res}_{z=a} \frac{x(z)^m}{dx(z)} (-2y(\frac{1}{z})dx(z)\omega_1^g(\frac{1}{z}) + \sum_{h=0}^g \omega_1^h(\frac{1}{z})\omega_1^{g-h}(\frac{1}{z}) + \omega_2^{g-1}(\frac{1}{z}, \frac{1}{z})) \\
&+ \frac{1}{4} \sum_{a=\pm 1} \operatorname{Res}_{z=a} \frac{x(z)^m}{dx(z)} (-2y(\frac{1}{z})dx(z)\omega_1^g(\frac{1}{z}) + \sum_{h=0}^g \omega_1^h(\frac{1}{z})\omega_1^{g-h}(\frac{1}{z}) + \omega_2^{g-1}(\frac{1}{z}, \frac{1}{z})) \\
&= \frac{1}{4} \sum_{a=\pm 1} \operatorname{Res}_{z=a} \frac{x(z)^m}{dx(z)} (P_0^g(x(z))dx^2(z)) \\
&= 0
\end{aligned}$$

Note that in the general proof for arbitrary n , the terms added in line two are more complex recursions with Bergmann kernel terms present, contributing extra residues. \square

Corollary 3. *For the plane curve $x = z + 1/z$, $y = y(z)$*

$$\begin{aligned}
& \mathcal{D}y(\mathcal{D}) \{mN_1^g(m)\}_{m=-1} = 0 \\
& (1 + \mathcal{D}^2)y(\mathcal{D}) \{mN_1^g(m)\}_{m=-1} = 0 \\
& (1 - \mathcal{D}^2)^2 y(\mathcal{D}) \{mN_1^g(m)\}_{m=-2} = 0
\end{aligned}$$

When $n = 0$ the dilaton equation is used to define F^g .

Proof of Corollary 2. We need to show that

$$F^g = \frac{1}{2-2g} (1 - \mathcal{D}^2)y(\mathcal{D}) \{N_1^g(m)\}_{m=-1}$$

for $g \geq 2$. The definition of the symplectic invariant uses the $n = 0$ version of the dilaton equation

$$(28) \quad \sum_{\alpha} \operatorname{Res}_{z=\alpha} \Phi(z) \omega_1^g(z) =: (2g-2)F^g$$

so we can simply apply the proof of equation (10) to the left hand side of this and the result follows. \square

5. EXAMPLES

This section serves a few purposes. It describes different interesting examples and gives small (g, n) polynomials in each case. These examples led to some of the general theorems in this paper and give checks of all of the theorems. It is difficult to calculate infinite families of invariants so this section also gives examples of symplectic invariants F^g (corresponding to $n = 0$) which are known for all g .

5.1. $y(z) = z$ **Branched covers and discrete surfaces.** The curve

$$(29) \quad C = \begin{cases} x = z + 1/z \\ y = z \end{cases}$$

gives rise to two rather different counts.

Expand the invariants ω_n^g of the curve $(x, y) = (z + 1/z, z)$ in x_i around $x_i = \infty$ for $i = 1, \dots, n$. Eynard and Orantin [2] show that this gives a generating function for counting connected orientable discrete surfaces of genus g with n polygonal faces and a marked edge on each face. The coefficient of $\prod x_i^{-(l_i+1)}$ in the expansion of ω_n^g counts the surfaces consisting of l_i -sided polygons, $i = 1, \dots, n$.

The associated polynomials N_n^g , obtained by expanding ω_n^g in z_i around $z_i = 0$, were shown in [8] to count connected topologically inequivalent genus g branched covers of S^2 branched over 0, 1 and ∞ with ramification (b_1, \dots, b_n) over ∞ , ramification $(2, 2, \dots, 2)$ over 1 and ramification greater than 1 at all points above 0. They are counted in such a way that each branched cover contributes one divided by the order of its group of automorphisms. Equivalently, they count surfaces with n polygonal faces of *lengths* b_1, \dots, b_n . Although this resembles the count above, it is quite different. The number $N_n^g(b_1, \dots, b_n)$ is presented in [7] in terms of counting lattice points inside integral convex polytopes depending on (b_1, \dots, b_n) which make up a cell decomposition of $\mathcal{M}_{g,n}$, the moduli space of genus g curves with n labeled points. A brief description follows.

Let $\mathcal{M}_{g,n}$ be the moduli space of genus g curves with n labeled points. The *decorated* moduli space $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$ equips the labeled points with positive numbers (b_1, \dots, b_n) [10]. It has a cell decomposition due to Penner, Harer, Mumford and Thurston

$$(30) \quad \mathcal{M}_{g,n} \times \mathbb{R}_+^n \cong \bigcup_{\Gamma \in \mathcal{F}at_{g,n}} P_\Gamma$$

where the indexing set $\mathcal{F}at_{g,n}$ is the space of labeled fatgraphs of genus g and n boundary components. A *fatgraph* is a graph Γ with vertices of valency > 2 equipped with a cyclic ordering of edges at each vertex. The cell decomposition (30) arises by the existence and uniqueness of meromorphic quadratic differentials with foliations having compact leaves, known as Strebel differentials which can be described via labeled fatgraphs with lengths on edges. Restricting this homeomorphism to a fixed n -tuple of positive numbers (b_1, \dots, b_n) yields a space homeomorphic to $\mathcal{M}_{g,n}$ decomposed into compact convex polytopes

$$P_\Gamma(b_1, \dots, b_n) = \{\mathbf{x} \in \mathbb{R}_+^{E(\Gamma)} \mid A_\Gamma \mathbf{x} = \mathbf{b}\}$$

where $\mathbf{b} = (b_1, \dots, b_n)$ and $A_\Gamma : \mathbb{R}^{E(\Gamma)} \rightarrow \mathbb{R}^n$ is the incidence matrix that maps an edge to the sum of its two incident boundary components.

When the b_i are positive integers the polytope $P_\Gamma(b_1, \dots, b_n)$ is an integral polytope and we define $N_\Gamma(b_1, \dots, b_n)$ to be its number of positive integer points. The weighted sum of N_Γ over all labeled fatgraphs of genus g and n boundary components is the piecewise polynomial [7]

$$N_n^g(b_1, \dots, b_n) = \sum_{\Gamma \in \mathcal{F}_{\text{at}_{g,n}}} \frac{1}{|\text{Aut}\Gamma|} N_\Gamma(b_1, \dots, b_n)$$

The top homogeneous degree terms of the polynomials $N_{n,k}^g$ (k even) representing N_n^g coincides with Kontsevich's volume polynomial [6]

$$V_n^g(b_1, \dots, b_n) = \sum_{\Gamma \in \mathcal{F}_{\text{at}_{g,n}}} \frac{1}{|\text{Aut}\Gamma|} V_\Gamma(b_1, \dots, b_n)$$

where $V_\Gamma(b_1, \dots, b_n)$ is the volume of $P_\Gamma(b_1, \dots, b_n)$ induced from the Euclidean volumes on $\mathbb{R}^{E(\Gamma)}$ and \mathbb{R}^n . Kontsevich showed that the volume polynomial is a generating function for intersection numbers over the moduli space, so this gives an alternative proof of Theorem 2 in this case.

Each integral point in the polytope $P_\Gamma(b_1, \dots, b_n)$ corresponds to a Dessin d'enfants defined by Grothendieck [5] which is a branched cover of S^2 branched over 0, 1 and ∞ with ramification (b_1, \dots, b_n) over ∞ , ramification $(2, 2, \dots, 2)$ over 1 and ramification greater than 1 at all points above 0 as defined above.

TABLE 1. $y = z$

g	n	# odd b_i	$N_n^g(b_1, \dots, b_n)$
0	3	0,2	1
1	1	0	$\frac{1}{48}(b_1^2 - 4)$
0	4	0,4	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 - 4)$
0	4	2	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 - 2)$
1	2	0	$\frac{1}{384}(b_1^2 + b_2^2 - 4)(b_1^2 + b_2^2 - 8)$
1	2	2	$\frac{1}{384}(b_1^2 + b_2^2 - 2)(b_1^2 + b_2^2 - 10)$
2	1	0	$\frac{1}{2^{16}3^{35}}(b_1^2 - 4)(b_1^2 - 16)(b_1^2 - 36)(5b_1^2 - 32)$

The description of N_n^g as counting lattice points inside cells of the moduli space enables one to prove $N_n^g(0, \dots, 0) = \chi(\mathcal{M}_{g,n})$ the orbifold Euler characteristic of the moduli space of genus g curves with n labeled points [7]. It was further shown in [8] that the dilaton equation together with vanishing properties of N_n^g proves $N_{g,n+1}(0, \dots, 0) = -(2g - 2 + n)N_{g,n}(0, \dots, 0)$. In particular, when $n = 0$ this gives $\chi(\mathcal{M}_{g,1}) = (2 - 2g)F^g$. Hence the symplectic invariants for $g > 1$ are given by the orbifold Euler characteristic of the moduli space of genus g curves

$$F^g = \chi(\mathcal{M}_g).$$

5.2. $y = \frac{1}{m}z^m$ **Monomials.** The example $y = z$ is the first in the family of examples

$$(31) \quad C = \begin{cases} x = z + 1/z \\ y = \frac{1}{m}z^m \end{cases}$$

In the following we demonstrate similar behaviour between the polynomials associated to the curve $y = \frac{1}{m}z^m$ for $m > 1$ and the $m = 1$ case, suggesting there may be an underlying enumeration problem. Unlike the other examples in this paper, there is not yet an interpretation of the polynomials coming from an enumeration problem.

TABLE 2. $y = \frac{1}{m}z^m$ for odd m

g	n	# odd b_i	$N_n^g(b_1, \dots, b_n)$
0	3	0,2	1
0	3	1,3	0
1	1	0	$\frac{1}{48}(b_1^2 - m^2 - 3)$
1	1	1	0
0	4	0,4	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 - m^2 - 3)$
0	4	1,3	0
0	4	2	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 - m^2 - 1)$
1	2	0	$\frac{1}{384}((b_1^2 + b_2^2)^2 - 4(m^2 + 2)(b_1^2 + b_2^2) + 3m^4 + 10m^2 + 19)$
1	2	1	0
1	2	2	$\frac{1}{384}(b_1^2 + b_2^2 - m^2 - 1)(b_1^2 + b_2^2 - 3m^2 - 7)$
2	1	0	$\frac{1}{2^{16}3^35}(5b_1^8 - (116m^2 + 196)b_1^6 + (834m^4 + 2476m^2 + 2402)b_1^4 - (2028m^6 + 7908m^4 + 13556m^2 + 13116)b_1^2 + 1305m^8 + 5628m^6 + 13494m^4 + 24636m^2 + 28665)$
2	1	1	0

Theorem 3 gives the following recursions relations which generalise the $m = 1$ case. They apply to all genus and can be used to determine the genus 0 and genus 1 polynomials.

$$\begin{aligned}
N_{n+1}^g(m, b_1, \dots, b_n) &= \sum_{j=1}^n \sum_{k=1}^{b_j} k N_n^g(b_1, \dots, b_n) |_{b_j=k} \\
(m+1)N_{n+1}^g(m+1, b_1, \dots, b_n) &+ (m-1)N_{n+1}^g(m-1, b_1, \dots, b_n) \\
&= 2m \sum_{j=1}^n \sum_{k=1}^{b_j} k N_n^g(b_1, \dots, b_n) |_{b_j=k} - m \sum_{j=1}^n b_j N_n^g(b_1, \dots, b_n)
\end{aligned}$$

$$\begin{aligned}
& (m-2)N_{n+1}^g(m-2, b_1, \dots, b_n) - 2mN_{n+1}^g(m, b_1, \dots, b_n) \\
& + (m+2)N_{n+1}^g(m+2, b_1, \dots, b_n) = m \sum_{j=1}^n \sum_{k=1 \pm b_j} k N_n^g(b_1, \dots, b_n)|_{b_j=k} \\
& N_{n+1}^g(m+1, b_1, \dots, b_n) - N_{n+1}^g(m-1, b_1, \dots, b_n) = m(2g-2+n)N_n^g(b_1, \dots, b_n)
\end{aligned}$$

TABLE 3. $y = \frac{1}{m}z^m$ for even m

g	n	# odd b_i	$N_n^g(b_1, \dots, b_n)$
0	3	0,2	0
0	3	1,3	1
1	1	0	0
1	1	1	$\frac{1}{48}(b_1^2 - m^2 - 3)$
0	4	0,4	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 - m^2)$
0	4	1,3	0
0	4	2	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 - m^2 - 2)$
1	2	0	$\frac{1}{384}(b_1^2 + b_2^2 - m^2)(b_1^2 + b_2^2 - 3m^2 - 8)$
1	2	1	0
1	2	2	$\frac{1}{384}((b_1^2 + b_2^2)^2 - 4(m^2 + 2)(b_1^2 + b_2^2) + 3m^4 + 8m^2 + 12)$
2	1	0	0
2	1	1	$\frac{1}{2^{16}3^35}(5b_1^8 - (116m^2 + 196)b_1^6 + (834m^4 + 2356m^2 + 1982)b_1^4 - (2028m^6 + 7428m^4 + 10796m^2 + 3396)b_1^2 + 1305m^8 + 5268m^6 + 10914m^4 + 11436m^2 + 1605)$

The following vanishing result is proved by considering the Euler characteristic of connected branched covers of the two-sphere.

Proposition 5.1 ([8]). *For $x = z + 1/z$ and $y = z$, if*

$$\sum_{i=1}^n b_i \leq -2\chi = 4g - 4 + 2n, \quad b_i > 0$$

then $N_n^g(b_1, \dots, b_n) = 0$.

In particular, $N_n^g(1, 1, \dots, 1) = 0$ which generalises as follows.

Proposition 5.2. *For $x = z + 1/z$ and $y = \frac{z^m}{m}$, $N_n^g(m, 1, 1, \dots, 1) = 0$.*

Proof. We have

$$N_n^g(m, b_1, \dots, b_n) = \sum_{j=1}^n \sum_{\substack{k=1 \\ k \not\equiv b_j(2)}}^{b_j} k N_n^g(b_1, \dots, b_n)|_{b_j=k}$$

As in Section 4.3 which makes sense of the string equation for $n = 0$, setting $b_1, \dots, b_n = 1$ leaves an empty sum on the right hand side, hence it is zero. \square

5.3. $y = \ln(z)$ **Partitions with Plancherel measure.** For a partition $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0$ define $|\lambda| = \sum \lambda_i$ and $n(\lambda) = \#\{i : \lambda_i \neq 0\}$. The Plancherel measure on partitions of size $|\lambda| = N$ uses the dimensions of irreducible representations of S_N , labeled by partitions λ , satisfying $\sum_{|\lambda|=N} \dim(\lambda)^2 = N!$. The partition function

$$Z_N(Q) = \sum_{n(\lambda) \leq N} \left(\frac{\dim \lambda}{|\lambda|!} \right)^2 Q^{2|\lambda|}$$

is related to the symplectic invariants of the curve [1]

$$(32) \quad C = \begin{cases} x = z + 1/z \\ y = \ln z \end{cases}$$

via the asymptotic expansion as $Q \rightarrow \infty$

$$\ln Z_N(Q) = \sum_g Q^{2-2g} F^g.$$

For $N \rightarrow \infty$, $\exp(-Q^2)Z_N(Q) \rightarrow 1$ so

$$F^g = \delta_{g,0}.$$

TABLE 4. $y = \ln z$

g	n	# odd b_i	$N_n^g(b_1, \dots, b_n)$
0	3	0,2	0
0	3	1,3	1
1	1	0	0
1	1	1	$\frac{1}{48}(b_1^2 - 3)$
0	4	0,4	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2)$
0	4	1,3	0
0	4	2	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 - 2)$
1	2	0	$\frac{1}{384}(b_1^2 + b_2^2 - 8)(b_1^2 + b_2^2)$
1	2	1	0
1	2	2	$\frac{1}{384}(b_1^2 + b_2^2 - 6)(b_1^2 + b_2^2 - 2)$
2	1	0	0
2	1	1	$\frac{1}{2^{16}3^35}(b_1^2 - 1)^2(5b_1^4 - 186b_1^2 + 1605)$

The curve (32) can be seen as the $m = 0$ case of the previous family of examples (31). The polynomials satisfy the following recursions relations.

$$N_{n+1}^g(0, b_1, \dots, b_n) = \sum_{j=1}^n \sum_{k=1}^{b_j} k N_n^g(b_1, \dots, b_n)|_{b_j=k}$$

$$N_{n+1}^g(1, b_1, \dots, b_n) = \sum_{j=1}^n \sum_{k=1}^{b_j} k N_n^g(b_1, \dots, b_n)|_{b_j=k} + \frac{\chi - |b|}{2} N_n^g(b_1, \dots, b_n)$$

for $\chi = 2 - 2g - n$ and $|b| = \sum_{j=1}^n b_j$.

In all calculated cases, for $x = z + 1/z$ and $y = z^m/m$ or $y = \ln z$, the genus 0 invariants N_n^0 take integral values. When $m = 1$, the genus 0 invariants are proven to be integral, and based on observation it seems reasonable to conjecture that the genus 0 invariants are integral for $m > 1$. This lends further evidence that there may be an underlying enumeration problem. In general for Gromov-Witten or moduli space calculations, the invariants are rational, and integral in genus 0.

5.3.1. $y = \ln(z) + cz$ *Expectation values of functions on partitions.* The following natural function on partitions

$$C_k(\lambda) = \sum_{i=1}^N \left(\lambda_i - i + \frac{1}{2} \right)^k - \left(-i + \frac{1}{2} \right)^k + (1 - 2^{-k}) \zeta(-k)$$

is called a Casimir in [1], and a shifted symmetric polynomial, notated by p_k , in [9]. The expectation values of C_2 with respect to the Plancherel measure on partitions are encoded in a generating function which gives rise to the following curve [1] (after making some coordinate changes.)

$$(33) \quad C = \begin{cases} x = z + 1/z \\ y = \ln z + cz \end{cases}, \quad c \text{ is a constant}$$

The curve has a symmetry

$$(z, c) \mapsto (-z, -c)$$

since $x \mapsto -x$ and $y \mapsto y + \ln(-1)$ which implies that when $n + k$ is even $N_{n,k}^g$ is a function of c^2 and when $n + k$ is odd $cN_{n,k}^g$ is a function of c^2 . In particular, F^g is a function of c^2 .

In the limit $c \rightarrow \infty$, $y(z)/c \rightarrow z$ (uniformly in neighbourhoods of ± 1) and this allows us to deduce that the symplectic invariants for $g > 1$ are asymptotic to the symplectic invariants for $y = z$, i.e.

$$F^g \sim \chi(\mathcal{M}_g) c^{2-2g}, \quad c \rightarrow \infty.$$

The polynomials interpolate between the $y = \ln z$ and $y = z$ cases. More precisely,

$$\lim_{c \rightarrow 0} N_n^g = N_n^g[y(z) = \ln z], \quad \lim_{c \rightarrow \infty} c^{2g-2+n} N_n^g = N_n^g[y(z) = z].$$

TABLE 5. $y = \ln z + cz$, $D = y'(1)y'(-1) = c^2 - 1$

g	n	# odd b_i	$N_n^g(b_1, \dots, b_n)$
0	3	0,2	$\frac{c}{D}$
0	3	1,3	$\frac{-1}{D}$
1	1	0	$\frac{c}{48D^2}(Db_1^2 - 4c^2 + 2)$
1	1	1	$\frac{1}{48D^2}(-Db_1^2 + 5c^2 - 3)$
0	4	0,4	$\frac{1}{4D^3}((c^4 - 1)(b_1^2 + b_2^2 + b_3^2 + b_4^2) - 4c^4)$
0	4	1,3	$\frac{-c}{2D^3}(D(b_1^2 + b_2^2 + b_3^2 + b_4^2) - 3c^2 + 1)$
0	4	2	$\frac{1}{4D^3}((c^4 - 1)(b_1^2 + b_2^2 + b_3^2 + b_4^2) - 2c^4 - 4c^2 + 2)$
1	2	0	$\frac{1}{384D^4}((c^2 + 1)D^2(b_1^2 + b_2^2)^2 + 32c^6 - 4D(3c^4 + 3c^2 - 2)(b_1^2 + b_2^2))$
2	1	0	$\frac{c}{2^{16}3^35D^7}(5(c^2 + 3)D^4b_1^8 - 8(39c^4 + 136c^2 - 59)D^3b_1^6 + 16(357c^6 + 1417c^4 - 1020c^2 + 254)D^2b_1^4 - 64(572c^8 + 2192c^6 - 1739c^4 + 806c^2 - 151)Db_1^2 + 6144c^7(12c^4 + 21c^2 + 2))$

5.3.2. *q-deformed partition.* The Plancherel weights $P(\lambda) = (\dim \lambda / |\lambda|!)^2$ used in Section 5.3 can be replaced by $P_q(\lambda) = (\dim_q \lambda / [|\lambda|]!)^2$, q -deformed Plancherel weights where q -numbers are used in place of integer combinatorial expressions. We will not give the details here since we will only use the spectral curve calculated in [1] for this case.

$$(34) \quad C = \begin{cases} x = (1 - \frac{z}{z_0})(1 - \frac{1}{zz_0}) \\ y = \frac{1}{x(z)} \left(-\ln z + \frac{1}{2} \ln \left(\frac{1 - \frac{z}{z_0}}{1 - \frac{1}{zz_0}} \right) \right) \end{cases},$$

The curve has a symmetry

$$(z, z_0) \mapsto (-z, -z_0)$$

which implies that when k is even $N_{n,k}^g$ is a function of z_0^2 and when k is odd $z_0 N_{n,k}^g$ is a function of z_0^2 . In particular, F^g is a function of z_0^2 . Put $z_0^2 = 1 - e^t$ so F^g is a function of e^t .

It was proven in [1] that

$$F^g = c_g + \sum_{d>0} |\chi(\mathcal{M}_g)| \frac{d^{2g-3}}{(2g-3)!} e^{-td}$$

for a constant c_g . The Euler characteristic $\chi(\mathcal{M}_g)$ arises in some sense by coincidence as an expression involving Bernoulli numbers, with no geometric relation to the moduli space. The asymptotics of F^g as $t \rightarrow 0$ can explain the appearance of $\chi(\mathcal{M}_g)$ geometrically. As $t \rightarrow 0$, or equivalently $z_0 \rightarrow 0$,

$$x \sim \frac{1}{z_0^2} - \frac{1}{z_0} \left(z + \frac{1}{z} \right) \equiv -\frac{1}{z_0} \left(z + \frac{1}{z} \right), \quad y \sim \frac{z_0^3}{2} \left(z - \frac{1}{z} \right) \equiv z_0^3 z$$

where x has been shifted by a constant and y has been adjusted by a linear function in x which does not affect the invariants. Under this scaling the invariants scale by

$$\omega_n^g \sim (-1)^{2-2g-n} z_0^{4-4g-2n} \omega_n^g[y = z]$$

and in particular

$$F^g \sim z_0^{4-4g} F^g[y = z] = |\chi(\mathcal{M}_g)| t^{2-2g}, \quad t \rightarrow 0$$

where $F^g[y = z] = \chi(\mathcal{M}_g)$ follows from the relation of the case $x = z + 1/z$, $y = z$ to counting lattice points in the moduli space of curves described in Section 5.1.

TABLE 6. q -deformed partitions

\mathbf{g}	\mathbf{n}	# odd b_i	$N_n^g(b_1, \dots, b_n)$
0	3	0,2	$\frac{-1}{z_0^2}(3z_0^2 + 1)$
0	3	1,3	$\frac{1}{z_0}(z_0^2 + 3)$
1	1	0	$\frac{-1}{48z_0^2}((3z_0^2 + 1)b_1^2 - 2z_0^2 - 4)$
1	1	1	$\frac{1}{48z_0}((z_0^2 + 3)b_1^2 - 3z_0^2 - 3)$
0	4	0,4	$\frac{1}{4z_0^4}((1 + z_0^2)(z_0^4 + 14z_0^2 + 1)(b_1^2 + b_2^2 + b_3^2 + b_4^2) - 4)$
0	4	1,3	$\frac{-1}{2z_0^3}((z_0^2 + 3)(3z_0^2 + 1)(b_1^2 + b_2^2 + b_3^2 + b_4^2) - z_0^4 + 4z_0^2 - 3)$
1	2	0	$\frac{1}{384z_0^4}((z_0^2 + 1)(z_0^4 + 14z_0^2 + 1)(b_1^2 + b_2^2)^2 - 32z_0^2 + 32 - 4(2z_0^6 + 3z_0^4 + 8z_0^2 + 3)(b_1^2 + b_2^2))$
2	1	0	$\frac{1}{2^{16}3^35z_0^6}(-5(3z_0^2 + 1)(3z_0^6 + 27z_0^4 + 33z_0^2 + 1)b_1^8 + (952z_0^8 + 224z_0^6 + 336z_0^4 + 3808z_0^2 + 312)b_1^6 + (-3680z_0^8 + 11360z_0^6 + 20688z_0^4 - 13952z_0^2 - 5712)b_1^4 + (-5696z_0^8 + 5120z_0^6 - 38592z_0^4 - 21248z_0^2 + 36608)b_1^2 - 36864(z_0^2 - 2)(z_0^2 - 1))$

5.4. $y = \frac{t}{t - \gamma z}$ **Discrete surfaces.** The example in Section 5.1 produced a generating function for counting connected orientable discrete surfaces of genus g with n polygonal faces and a marked edge on each face. That is a special case of more general connected orientable discrete surfaces of genus g with n marked polygonal faces—each containing a marked edge—together with unmarked faces of perimeter greater than 2, with n_i of perimeter i for $i \geq 3$. A spectral curve for this problem is constructed in [2], where the expansion of ω_n^g in x around $x = \infty$ gives a generating function that counts these discrete surfaces. The coefficient of $\prod x_i^{-(l_i+1)} t_3^{n_3} \dots t_d^{n_d}$ in the expansion of ω_n^g counts the surfaces consisting of t_3 (unmarked) triangles, t_4 squares, ... and l_i -sided marked polygons, $i = 1, \dots, n$. If we set $t_k = -1$ for all $k \geq 3$ then the spectral curve is given by

$$(35) \quad C = \begin{cases} x = -2t + \gamma(z + 1/z) \\ y = \frac{t}{t - \gamma z} \end{cases}, \quad \gamma^2 = t(t + 1)$$

and the generating function gives an alternating sum of numbers of discrete surfaces.

The curve has a symmetry

$$(z, \gamma) \mapsto (-z, -\gamma)$$

which implies that when k is even $N_{n,k}^g$ is a function of γ^2 and when k is odd $\gamma N_{n,k}^g$ is a function of γ^2 . In particular, F^g is a function of γ^2 , hence a function of t as shown below.

The curve also has a less obvious symmetry

$$(t, \gamma) \mapsto (-t-1, -\gamma)$$

which takes $x \mapsto 2-x$, and

$$y \mapsto \frac{t+1}{t+1-\gamma z} = \frac{\gamma^2}{\gamma^2-t\gamma z} = 1 + \frac{tz}{\gamma-tz} \sim 1 + \frac{tz^{-1}}{\gamma-tz^{-1}} = 1-y$$

where the \sim sign corresponds to reparametrising the curve by $z \mapsto z^{-1}$. This implies that under this symmetry N_n^g is invariant, respectively skew invariant, when n is even, respectively n is odd. In particular, F^g is invariant under $t \mapsto -t-1$.

The symplectic invariants of this curve conjecturally use the $n=0$ case of the enumerative problem above—an alternating count of surfaces consisting of t_3 triangles, t_4 squares, ... and no marked polygons. The solution of the $n=0$ problem is

$$\sum_{k>0} \chi(\mathcal{M}_{g,k}) \frac{t^k}{k!}$$

where $\chi(\mathcal{M}_{g,k})$ is the orbifold Euler characteristic of the moduli space of genus g curves with k labeled points. This uses a cell decomposition of the moduli space where each polygon labels a cell. The alternating sum over all cells gives the Euler characteristic $\chi(\mathcal{M}_{g,k})/k!$ of the moduli space of genus g curves with k (unlabeled) points.

As $t \rightarrow 0$,

$$x \sim t^{1/2}(z+1/z), \quad y \sim -t^{1/2}/z$$

hence the curve behaves asymptotically like $x = z + 1/z$, $y = -1/z$, or equivalently $y = z(-1/z + x)$ which is the example in Section 5.1 with $F^g[y=z] = \chi(\mathcal{M}_g)$. Hence $F^g \sim \chi(\mathcal{M}_g)t^{2-2g}$ as $t \rightarrow 0$. Adding the two contributions gives an asymptotic formula which is conjecturally exact:

$$\begin{aligned} F^g &= \chi(\mathcal{M}_g)t^{2-2g} + \sum_{k>0} \chi(\mathcal{M}_{g,k}) \frac{t^k}{k!} \\ &= \chi(\mathcal{M}_g) \left(t^{2-2g} + (-1)^k \binom{2g+k-3}{2g-3} t^k \right) \\ &= \chi(\mathcal{M}_g) (t^{2-2g} + (t+1)^{2-2g}) \end{aligned}$$

where $\chi(\mathcal{M}_g) = \zeta(1-2g)/(2-2g)$.

It is interesting that the symmetry $t \mapsto -t-1$ of F^g (proved *a priori*) suggests that the asymptotic part t^{2-2g} determines, and is determined by, the solution of the $n=0$ enumerative problem, $(t+1)^{2-2g}$.

5.4.1. *Relation to counting surfaces.* As discussed above, the expansion in x of ω_n^g around $x = \infty$ gives a generating function for counting discrete surfaces, where the coefficient of $\prod x_i^{-(l_i+1)} t_3^{n_3} \dots t_d^{n_d}$ in the expansion counts surfaces consisting of t_3 (unmarked) triangles, t_4 squares, ... and l_i -sided marked polygons, $i = 1, \dots, n$. We can find this coefficient explicitly by evaluating

$$\begin{aligned}
T_{l_1, \dots, l_n}^g &:= (-1)^n \operatorname{Res}_{x_1=\infty} \dots \operatorname{Res}_{x_n=\infty} x_1^{l_1} \dots x_n^{l_n} \omega_n^g \\
&= (-1)^n \prod_{i=1}^n \operatorname{Res}_{z_i=\infty} \gamma^{l_i} \left(z_i + \frac{1}{z_i} \right)^{l_i} \sum_{b_1, \dots, b_n=1}^{\infty} N_n^g(b_1, \dots, b_n) \prod_{i=1}^n b_i z_i^{b_i-1} dz_i \\
&= (-1)^n \prod_{i=1}^n \operatorname{Res}_{z_i=0} \gamma^{l_i} \left(\frac{1}{z_i} + z_i \right)^{l_i} \sum_{b_1, \dots, b_n=1}^{\infty} N_n^g(b_1, \dots, b_n) \prod_{i=1}^n b_i z_i^{b_i-1} dz_i \\
&= (-1)^n \prod_{i=1}^n \operatorname{Res}_{z_i=0} \gamma^{l_i} \sum_{k_1, \dots, k_n=0}^{l_i} \sum_{b_1, \dots, b_n=1}^{\infty} N_n^g(b_1, \dots, b_n) \prod_{i=1}^n b_i \binom{l_i}{k_i} z_i^{l_i-2k_i+b_i-1} dz_i \\
&= (-1)^n \gamma^{\sum l_i} \sum_{k_i > \frac{l_i}{2}}^{l_i} N_n^g(2k_1 - l_1, \dots, 2k_n - l_n) \prod_{i=1}^n (2k_i - l_i) \binom{l_i}{k_i}
\end{aligned}$$

where the N_n^g 's associate to the spectral curve computed for particular choices of t_k . In our example, we have set $t_3 = t_4 = \dots = -1$, so that T_{l_1, \dots, l_n}^g counts the bias of discrete surfaces having even or odd numbers of faces:

$$T_{l_1, \dots, l_n}^g = \sum_{v=1}^{\infty} t^v \sum_{S \in M_n^g(v, l_1, \dots, l_n)} \frac{(-1)^{\# \text{ unmarked polygons in } S}}{|Aut(S)|}$$

where $M_n^g(v, l_1, \dots, l_n)$ is the set of connected, orientable, discrete surfaces of genus g , constructed by connecting n marked polygons of perimeter l_1, \dots, l_n and any finite number of unmarked polygons using only v vertices. From [2] this is a finite set. For example,

$$T_l^{(2)} = \begin{cases} 0 & 0 \leq l \leq 4 \\ -8t - 8t^2 & l = 5 \\ 36t + 108t^2 + 72t^3 & l = 6 \\ -49t - 490t^2 - 882t^3 - 441t^4 & l = 7 \end{cases}$$

Remark. It seems that T_{l_1, \dots, l_n}^g is polynomial in t . Equivalently, most of the coefficients of powers of t vanish. This means that besides finitely many exceptions for a small number of vertices, there is a duality between discrete surfaces with an even and odd number of faces. Furthermore, the following vanishing result shows that there are no exceptions when the b_i are positive and small enough.

$$N_n^g(b_1, \dots, b_n) = 0 \quad \text{for} \quad \sum_{i=1}^n b_i < 2g + n, \quad b_i > 0.$$

The vanishing result is equivalent to the fact that $\omega_n^g(z_1, \dots, z_n)$ vanishes at $z_i = 0$ with homogeneous degree $2g$ which can be proved inductively from the recursive definition. A similar vanishing result holds for the $y = z$ case—see Proposition 5.1.

TABLE 7. Discrete surfaces

g	n	# odd b_i	$N_n^g(b_1, \dots, b_n)$
0	3	0,2	$\frac{1+2t}{\gamma^2}$
0	3	1,3	$\frac{-2}{\gamma}$
1	1	0	$\frac{1+2t}{48\gamma^2}(b_1^2 - 4)$
1	1	1	$\frac{-1}{24\gamma}(b_1^2 - 1)$
0	4	0,4	$\frac{1}{4\gamma^4}((8\gamma^2 + 1)(b_1^2 + b_2^2 + b_3^2 + b_4^2) - 8\gamma^2 - 4)$
0	4	1,3	$\frac{-1}{\gamma}(1 + 2t)(b_1^2 + b_2^2 + b_3^2 + b_4^2 - 1)$
0	4	2	$\frac{1}{4\gamma^4}((8\gamma^2 + 1)(b_1^2 + b_2^2 + b_3^2 + b_4^2) - 8\gamma^2 - 2)$
1	2	0	$\frac{1}{384\gamma^4}(b_1^2 + b_2^2 - 4)((8\gamma^2 + 1)(b_1^2 + b_2^2) - 16\gamma^2 - 8)$
1	2	1	$\frac{-1}{96\gamma^3}(1 + 2t)(b_1^2 + b_2^2 - 1)(b_1^2 + b_2^2 - 5)$
1	2	2	$\frac{1}{384\gamma^4}(b_1^2 + b_2^2 - 2)((8\gamma^2 + 1)(b_1^2 + b_2^2) - 32\gamma^2 - 10)$
2	1	0	$\frac{1+2t}{2^{16}3^35\gamma^6}(b_1^2 - 4)(b_1^2 - 16)((80\gamma^2 + 5)b_1^4 - (608\gamma^2 + 212)b_1^2 + 1152\gamma^2 + 1152)$
2	1	1	$\frac{-1}{2^{15}3^35\gamma^5}(b_1^2 - 1)(b_1^2 - 9)((80\gamma^2 + 15)b_1^4 - (1408\gamma^2 + 438)b_1^2 + 3632\gamma^2 + 1575)$

5.4.2. *Quadrangulations.* One can enumerate discrete surfaces consisting of quadrilaterals by setting $t_k = 0$ for $k \neq 4$ in the enumeration of discrete surfaces defined in Section 5.4 counts. (There we set $t_k = -1$ for $k \geq 3$.) The spectral curve for this problem is given in [2]

$$(36) \quad C = \begin{cases} x = z + 1/z \\ y = tz - t_4\gamma^4 z^3 \end{cases}, \quad \gamma^2 = \frac{1 - \sqrt{1 - 12tt_4}}{6t_4}$$

In this case the recursion relations between polynomials are

$$\begin{aligned}
tN_{n+1}^g(1, b_S) - 3t_4\gamma^4 N_{n+1}^g(3, b_S) &= \sum_{j=1}^n \sum_{k=1}^{b_j} k N_n^g(b_S)|_{b_j=k} \\
2(t - t_4\gamma^4)N_{n+1}^g(2, b_S) - 4t_4\gamma^4 N_{n+1}^g(4, b_S) \\
&= 2 \sum_{j=1}^n \sum_{k=1}^{b_j} k N_n^g(b_S)|_{b_j=k} - \sum_{j=1}^n b_j N_n^g(b_S)
\end{aligned}$$

$$(t - t_4\gamma^4)N_{n+1}^g(1, b_S) + 3(t - 2t_4\gamma^4)N_{n+1}^g(3, b_S) \\ + 5t_4\gamma^4 N_{n+1}^g(5, b_S) = \sum_{j=1}^n \sum_{k=1 \pm b_j} k N_n^g(b_S)|_{b_j=k}$$

$$-tN_{n+1}^g(0, b_S) + (t + t_4\gamma^4)N_{n+1}^g(2, b_S) - t_4\gamma^4 N_{n+1}^g(4, b_S) = (2g - 2 + n)N_n^g(b_S)$$

where $b_S = (b_1, \dots, b_n)$.

In the following table $y'(1) = t - 3t_4\gamma^4$.

TABLE 8. Quadrangulations

g	n	# odd b_i	$N_n^g(b_1, \dots, b_n)$
0	3	0,2	$\frac{1}{y'(1)}$
0	3	1,3	0
1	1	0	$\frac{1}{48y'(1)^2}(y'(1)b_1^2 + 36t_4\gamma^4 - 4t)$
1	1	1	0
0	4	0,4	$\frac{1}{4y'(1)^3}(y'(1)(b_1^2 + b_2^2 + b_3^2 + b_4^2) + 36t_4\gamma^4 - 4t)$
0	4	1,3	0
0	4	2	$\frac{1}{4y'(1)^3}(y'(1)(b_1^2 + b_2^2 + b_3^2 + b_4^2) + 30t_4\gamma^4 - 2t)$
2	1	0	$\frac{1}{2^{15}3^35y'(1)^7}(5y'(1)^4b_1^8 - 24y'(1)^3(13t - 155t_4\gamma^4)b_1^6 \\ + 48y'(1)^2(119t^2 - 2090t_4\gamma^4t + 17295t_4^2\gamma^8)b_1^4 \\ - 256y'(1)(143t^3 - 2793t^2t_4\gamma^4 + 25857tt_4\gamma^8 - 237735t_4^3\gamma^{12})b_1^2 \\ + 73728t^4 - 1548288t_4\gamma^4t^3 + 13934592t_4^2\gamma^8t^2 \\ - 36495360tt_4^3\gamma^{12} + 1134673920t_4^4\gamma^{16})$
2	1	1	0

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